On the large-scale structure of the inflationary universe

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Abstract

We study the evolution of quantum fluctuations of a scalar field which is coupled to the geometry, in an exponentially expanding universe. We derive an expression for the spectrum of intrinsic perturbations, and it is shown that the intrinsic degrees of freedom are well behaved in the infra-red part of the spectrum. We conclude that quantum fluctuations do not change the dynamics of the spacetime in a way which makes its evolution non-perturbative and stochastic. This result contradicts previous derivations which are based on the study of a quantum field on a fixed geometry.

1 Introduction

It is by now widely believed that the universe has gone through an early period of nearly exponential expansion (see e.g. [1] for an overview). Apart from solving the horizon, flatness and monopole problems, the assumption of nearly exponential expansion has the important implication of producing a nearly scale independent spectrum of perturbations, which is in agreement with observations [2]. Quantum fluctuations of the field which drives inflation, the inflaton, are generally believed to be the source of the perturbations which gave rise to the observed structure in the universe at late times. An idea which goes further has been proposed by Linde [3] - [5], who argues that quantum fluctuations of the inflaton grow in amplitude and decohere during inflation. Due to decoherence, the state of the inflaton can be described by a probability distribution, and due to quantum fluctuations, the time evolution of the inflaton becomes stochastic. Although the mean field of the inflaton tends to evolve to a minimum of its potential, the spread of the probability distribution in time allows for a motion away from the minimum of the potential, with some finite probability. If

there is a sufficiently large probability of finding the inflaton at a larger value of the potential, then inflation could proceed indefinitely in certain regions of the spacetime, which then dominate in terms of the spatial volume due to the expansion, while inflation may have come to an end in other regions. The large-scale structure of the spacetime can then be expected to be very irregular, hence the name 'stochastic inflation'. In this paper we will reconsider Linde's idea of stochastic inflation, by studying the evolution of a quantized scalar field which drives inflation on a *perturbed* rather than a fixed geometry. The important difference between our derivation and the derivations in [3] - [5] arises from the definition and the interpretation of the perturbations. Since in previous treatments one considered inflaton perturbations on a fixed geometry, it was possible to define a probability for finding the inflaton field in a specific state at some time. This picture breaks down when one considers quantum fluctuations of the inflaton field, which are coupled to the geometry, since then there is no notion of a point in space and time at which one can compare perturbations of the inflaton field. Instead, one has to deal with the intrinsic perturbations of the inflaton and the geometry, on a background geometry which can be taken to be a Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. If the intrinsic perturbations of the inflaton and the geometry tend to become nonperturbatively large at late times or large distances, then this shows that the spacetime evolves away from FLRW, as is the case in the stochastic inflation picture. We will show that this does not happen when the coupling between the inflaton and the geometry perturbations is taken into account.

The paper is structured as follows. In section 2 we discuss the generation of long-range quantum correlations of a scalar field, in an unperturbed exponentially expanding FLRW spacetime. It appears that these quantum correlations diverge in the limit where the mass of the field tends to zero. This effect can be explained from two different viewpoints; in comoving coordinates we find that individual quantum fluctuations 'freeze out', while in a static coordinatization the quantum field appears to 'heat up'.

In section 3 we discuss the issue of decoherence, and the generation of classical perturbations from quantum perturbations.

The quantization of the inflaton and geometry perturbations is considered in section 4, where we apply Bardeen's approach [7] to describe the intrinsic degrees of freedom of the system. The spectrum of intrinsic quantum perturbations is obtained in the idealized case of vanishing inflaton mass and exact exponential expansion, and we study the nature of the geometry perturbations at late times and large length scales.

It is shown that the intrinsic deviation of the perturbed spacetime from FLRW does not grow at large length scales or at late times. This observation, and the observational bound on the amplitude of the perturbations, justify our perturbative approach to the problem. We conclude in section 5 with the observation that no nonperturbative effects occur of the kind which could drive stochastic inflation.

From now on we choose our units so that $\hbar = c = 8\pi G = 1$, and we let Greek indices run from 0 to 3, while Latin indices run from 1 to 3.

2 The generation of fluctuations

In this section we will consider the generation of quantum fluctuations in an exponentially expanding FLRW spacetime, and its analytic extension which is de Sitter spacetime. The use of exactly exponentially expanding FLRW has the advantage that there are exact results available (see e.g. [9] [10]), which is not the case for more general inflating spacetimes. Although de Sitter spacetime and a nearly exponentially expanding FLRW spacetime are topologically very different, there are good reasons to believe that locally defined physical quantities are approximately the same in these two spacetimes. This observation is known as the 'no hair conjecture' for exponentially expanding spacetimes [13]. The metric of spatially flat exponentially expanding FLRW in terms of comoving coordinates reads,

$$\mathrm{d}s^2 = (H\eta)^{-2}\ (-\mathrm{d}\eta^2 + \delta_{ij}\mathrm{d}\mathbf{x}^i\mathrm{d}\mathbf{x}^j), \tag{1}$$

where $\eta \in \mathbf{R}^-$ denotes the conformal time parameter, and H is the Hubble constant. Let us consider a free Klein-Gordon field ϕ with a mass m in the geometry (1). In the case where the quantum state of the ϕ-field is invariant under spatial translations, the field operator which describes the quantized ϕ-field can be expanded in terms of a basis of modes which are the product of normalized plane waves Q, which we define in the appendix, and a time dependent factor ϕ_0 which solves the equation

$$\ddot{\phi}_0 - 2\eta^{-1}\dot{\phi}_0\ + (\mathbf{k}^2 + (\frac{m}{H\eta})^2)\phi_0 = 0, \tag{2}$$

where a dot denotes differentiation with respect to the conformal time η, and $\mathbf{k}$ denotes the comoving wavenumber. In the case where the mass m of the field is zero, the solutions of equation (2) are given by,

$$\phi_0 = c\eta^{\frac{3}{2}} H^{(1)}_{\frac{3}{2}}(-\mathbf{k}\eta), \tag{3}$$

where $H^{(1)}_q$ denotes the Hankel function of the first kind, and c is a normalization constant [24]. At late times, i.e. when $\eta \uparrow 0$, the massless solutions (3) approach a nonzero constant, while massive solutions of equation (2) go to zero as $(-\eta)^{\frac{m^2}{3H^2}}$. Note that although the amplitude of the modes remains large at late times when $m \ll H$, the probability density $j_0 = \frac{H\eta}{2i}(\phi^*\dot{\phi} - \dot{\phi}^*\phi)$ which is associated with the modes decreases as fast as η^3. The dynamical picture of the quantum fluctuations in an exponentially expanding FLRW spacetime is that their *amplitude* remains large at late times, in spite of exponentially vanishing

probability densities. In terms of the quantization which is natural in comoving coordinates, there is no change in the occupation number of different modes, and for this reason there is no creation of particles [12].

Let $G_0(\mathbf{x})$ denote the two point correlation function of the massless ϕ-field which is assumed to be in the vacuum state which is invariant under spatial translations. The correlation function $G_0(\mathbf{x})$ is given by a sum over modes, which factorize in terms of the time dependent solutions ϕ_0, and the spatially dependent harmonics Q (see e.g. [12]),

$$G_0(\mathbf{x}) = \frac{1}{2}\int \mathrm{d}^3\mathbf{k}|\phi_0(\eta,\mathbf{k})|^2 \; Q^*(\mathbf{k},\mathbf{x})Q(\mathbf{k},0). \tag{4}$$

Let $G_0(\mathbf{k})$ denote the spatial Fourier transform of $G_0(\mathbf{x})$, defined by expression (36) in the appendix. From expression (4) it follows directly that

$$G_0(\mathbf{k}) = \frac{1}{2}|\phi_0(\eta,\mathbf{k})|^2. \tag{5}$$

Using expression (3) for ϕ_0, and the asymptotic behavior of the Hankel function, $H_q^{(1)}(z) \sim z^{-q}$ for $z \downarrow 0$, we find the asymptotic expression for $G_0(\mathbf{k})$,

$$G_0(\mathbf{k}) \sim \mathbf{k}^{-3}, \tag{6}$$

for $\mathbf{k} \downarrow 0$. Expression (6) cannot be integrated about $\mathbf{k} = 0$, and therefore the correlation function $G_0(\mathbf{x})$, which is formally defined by expression (5), diverges for arbitrary $\mathbf{x}$. This is the well known infra-red divergence of the correlation function for a massless scalar field in an exponentially expanding FLRW spacetime [8] - [11].

If one is interested in the time dependence of quantum correlations, then equation (6) is somewhat difficult to interpret, since the comoving wavenumber $\mathbf{k}$, and the physical wavenumber k, are related by the time dependent expression $k = H\eta\mathbf{k}$. In order to describe the time dependence of quantum correlations, it is natural to consider the Fourier transform of the correlation function (4) with respect to harmonics which are normalized in *physical* k-space. From expression (5) and the relation (41) in the appendix, it follows that the Fourier transform of $G_0(\mathbf{x})$ in physical k-space is given by

$$G_0(k) = (H\eta)^{-3}G_0(\mathbf{k}). \tag{7}$$

The asymptotic behavior of $G_0(k)$ follows from expression (6) and (7), i.e.,

$$G_0(k) \sim k^{-3}, \tag{8}$$

for $k \downarrow 0$ and η constant, and there is no time dependence in this equation.

The infra-red divergence in equation (8) is related to the global properties of the exponentially expanding FLRW geometry. If one assumes that the exponential expansion has started at a finite time in the past, then one can show

that $G_0(k)$ is well behaved about $k = 0$ [8]. This follows e.g. by calculating the infra-red contribution to the expectation value of $|\phi|^2$,

$$\langle|\phi|^2\rangle = \int_0^\Lambda \mathrm{d}^3k \; G(k), \tag{9}$$

where Λ is an upper boundary on the k-space integral. It can be shown that $\langle|\phi|^2\rangle$ grows linearly in time when the exponential expansion has started at a finite time in the past, and $\langle|\phi|^2\rangle$ approaches a constant for a positive mass m [8].

A different perspective from which one can look at the quantum correlations in an exponentially expanding FLRW spacetime, is the presence of a thermal distribution of particles which is natural with respect to a static coordinate system. On the analytic extension of an exponentially expanding FLRW spacetime, which is de Sitter spacetime, one can choose coordinates so that the metric on a section of the spacetime takes the static form

$$\mathrm{d}s^2 = H^{-2}(-\sin^2\theta \mathrm{d}t^2 + \mathrm{d}\theta^2 \;\; + \cos^2\theta \mathrm{d}\Omega^2), \tag{10}$$

where $\theta \in [0, \pi/2]$ is related to the commonly used radial coordinate r by $r = H^{-1}\cos\theta$, and $\mathrm{d}\Omega^2$ is the surface element on the unit two-sphere. We may now define quanta of the ϕ-field, for which the wave function $\phi_{\omega lm}$ factorizes in terms of a time dependent plane wave $(2\omega)^{-\frac{1}{2}}e^{-i\omega t}$, a two-sphere harmonic Y_{lm}, and a function of θ and ω, l, m. Gibbons and Hawking [13] showed that when the quantum state of the field ϕ is analytic at the horizon $\theta = 0$, then quanta with wave functions $\phi_{\omega lm}$ are present with nonzero occupation numbers,

$$n_{\omega lm} = (e^{2\pi\omega/H} - 1)^{-1}. \tag{11}$$

In a spatially flat FLRW spacetime, a thermal distribution of the form (11) does not give rise to divergent long range correlations, even for $m = 0$, since in this case there is a phase space element of the form d^3k which cancels the divergence in expression (11) at $\omega = 0$. In the quantization which is based on a static coordinatization of de Sitter spacetime, the situation is however different since a constant time section in the metric (10) is one half of a three-sphere. The phase space in which the distribution function (11) is evaluated consists therefore of the continuous variable ω, and the discrete labels l, m.

Let us now consider the massless case where $m = 0$, and $\omega \in \mathbf{R}^+$. The infra-red contribution to the vacuum expectation value of $|\phi|^2$, is obtained by taking the integral over ω about $\omega \approx 0$, of the occupation number, multiplied by the squared modulus of the wave function. Indeed, the occupation number $n_{\omega 00}$ and the square of $\phi_{\omega 00}$ diverge as ω^{-1} near $\omega = 0$, and their formal integral over ω diverges. Recall that in the quantization which is based on comoving coordinates we found an infra-red divergent vacuum expectation value of $|\phi|^2$, which is due to the 'freezing' of zero point fluctuations, while no particles are created. In the

quantization which is based on static coordinatization, the vacuum expectation value of $|\phi|^2$ is also infra-red divergent, but this time it is due to the 'heating' of the quantum field on spatially bounded sections.

3 Classical and quantum perturbations

In the previous section, we discussed the generation of quantum correlations in an exponentially expanding FLRW spacetime. In the quantization which is based on comoving coordinates, we found quantum fluctuations with large amplitudes, although in this description the occupation number remains zero for each mode [12]. The assumption that large amplitudes imply large occupation numbers seems to play a role in a number of references [3] - [6], where it is used to argue that long range correlations become classical in some sense.

The question how quantum correlations at early times evolve into classical perturbations at late times, appears to receive little attention in the literature, although efforts towards a more satisfactory description have been made (see e.g. [15] - [19]). Without going into much detail, let us go through some of the ideas which play a role in the transition from quantum to classical perturbations. What appears to be crucial, is the *decoherence* of inequivalent histories [20]. In the context of the discussion which is given in the previous section, a history $\phi(x)$ is a configuration of the ϕ-field which is given over the entire spacetime. In physically relevant situations, it is often the case that only some degrees of freedom of the ϕ-field are accessible to observation, and it is natural to consider the equivalence classes of histories which cannot be distinguished by observation. While the state of the ϕ-field is generally given by a density matrix, a 'reduced density matrix' can be obtained by projecting the density matrix onto sets of inequivalent histories. Decoherence occurs when the reduced density matrix is approximately diagonal, and in this case the diagonal elements can be interpreted as approximate probabilities for inequivalent histories. When two or more sets of histories are mutually exclusive at a classical level (e.g., as in Schrödinger's experiment where there are two states of a cat in a box), then they must have decohered to a high degree of accuracy, in order to account for the absence of observed interference effects.

Note that there is a considerable amount of freedom to choose a criterion by which histories are considered to be equivalent. Which criterion is natural in a certain situation, depends on the type of measurement which is conducted.

As an example, let us consider again the case of a free scalar field ϕ, which is in the translation invariant vacuum state in an exponentially expanding FLRW spacetime. A geodesic observer in this spacetime perceives an event horizon at the Hubble distance H^{-1}, and can only respond to quanta which originate from somewhere within this event horizon. One can show that a geodesic observer responds to quanta which have a positive frequency with respect to the observers proper time (see e.g. [13]). These quanta are naturally defined in terms of a

quantization which is based on a static coordinate system of the form (10), where the coordinates are chosen so that the center of the coordinate system at $\theta = \frac{\pi}{2}$ coincides with the worldline of the geodesic observer. When expressed in terms of the quantization which is based on static coordinates, the state of the ϕ-field is described by a reduced density matrix, which appears to be completely diagonal [13]. Decoherence is in this case obtained by summing over the degrees of freedom of the ϕ-field which are outside the event horizon. However, the notion of an event horizon is observer dependent, and the reduced density matrix is in this case only natural to describe the measurement of one specific geodesic observer.

In the case of the early universe, there are no observers which can absorb quanta of a scalar field, and one has to consider what one could call a 'measurement situation' [14] [20]. Essentially this means that one introduces a coupling of the ϕ-field with itself or other fields (for instance by adding a cubic interaction term in the Lagrangian, or by considering the nonlinear coupling between the ϕ-field and gravitational degrees of freedom). The effect of an interaction term is that quantum fluctuations at large length scales (compared to the Hubble radius H^{-1}), are now coupled to quantum fluctuations at small length scales. By summing over perturbations of the ϕ-field with a wavelength which is small compared to the Hubble radius, one obtains a reduced density matrix which describes the decohered large scale fluctuations of the ϕ-field [15]-[19].

Let us now assume that decoherence occurs, so that one can define a probability distribution $P(\phi, t)$ which describes the probability P of finding the quantum field with a value ϕ at a point in the spacetime with time t. In Linde's approach to stochastic inflation [3]-[5], the time evolution of the probability distribution $P(\phi, t)$ is described by a diffusion equation,

$$\frac{\partial P(\phi,t)}{\partial t} = D\frac{\partial^2 P(\phi,t)}{\partial^2 \phi}, \tag{12}$$

where D is a positive constant. The spread of the quantum field ϕ about its mean value $\langle\phi\rangle$, is given by,

$$\langle|\phi^2|\rangle = \int \mathrm{d}\phi|\phi - \langle\phi\rangle|^2 \; P(\phi,t). \tag{13}$$

Recall that in section 2 we calculated $\langle|\phi^2|\rangle$ for the case of a free field in exponentially expanding FLRW. Although the expression was found to be divergent in the case of exact exponential expansion, a linear growth in time can be found when the exponential expansion has started at a finite time in the past [9]. This result can be used to determine the value of the diffusion constant D in expression (12) (see e.g. [4]).

An important limitation of the derivation which we sketched above, is that the probability distribution $P(\phi, t)$ is defined for an FLRW geometry. However, since we are interested in perturbations at length scales large compared to the

Hubble radius, there is no good reason why one can ignore the coupling between the inflaton perturbations and the geometry [7]. But if one does consider the coupling between the inflaton perturbations and the geometry, then we do not have a single geometry. Instead, for every configuration of the inflaton, one has a different geometry, and there is no notion of a single point in space and time for which one can define a probability $P(\phi, t)$ for finding the field with a value ϕ at time t. Indeed, one can show that quantum fluctuations of the inflaton field can be interpreted equally well as quantum fluctuations of only the metric, in a coordinate system where there is no inflaton perturbation at all.

In the following section, we derive the spectrum of *intrinsic* perturbations of the inflaton and the geometry which are generated during inflation.

4 Quantum fluctuations and stochastic inflation

In the previous sections, we studied the amplification of quantum fluctuations of a scalar field in an exponentially expanding FLRW spacetime. The correlation function was found to be infra-red divergent in the case of a massless scalar field, while a finite result was found for a positive scalar field mass or when we assumed that the exponential expansion had started at a finite time in the past. In the physically more realistic case of a universe where approximate exponential expansion is driven by a scalar field with a potential term, the scalar field mass will generally be nonzero and time dependent, and the expansion will tend to slow down gradually before inflation ends. However, since our aim is to investigate the possibility of an unbounded growth of intrinsic perturbations during inflation, we will neglect the mass of the inflaton and the slowing down of the expansion while the inflaton moves down the potential. In our perturbative approach to describe the evolution of quantum fluctuations of the inflaton and the geometry during inflation, we will therefore adopt a background geometry of the form (1). As is usual in cosmology, we define a perturbation of a physical quantity as the difference of the same physical quantity, evaluated at corresponding points in a perturbed and a background spacetime. What one calls a perturbation therefore depends on a mapping between points in a perturbed and a background spacetime, which is called a gauge. In the following we will use Bardeen's formalism [7] to deal with the degrees of freedom which, to linear order, do not depend on the choice of gauge. In our description of a scalar field in a perturbed spatially flat FLRW spacetime, it will be sufficient to consider only scalar perturbations, and the metric can be expanded as

$$g_{\mu\nu} = S^2 \sum_{\mathbf{k}} \Big[-\delta^0_\mu \delta^0_\nu (1+2A)Q \;\; + (\delta^0_\mu \delta^i_\nu + \delta^i_\mu \delta^0_\nu) B Q_{;i} \;\; + h_{\mu\nu}(1+2H_L Q) \\ +2H_T h^i_\mu h^j_\nu \; (\mathbf{k}^{-2} Q_{;i;j} + \frac{1}{3}\delta_{ij} Q) \Big] , \tag{14}$$

where $;i$ denotes the derivative with respect to the comoving spatial coordinate $\mathbf{x}^i$, $h^\mu_\nu := \delta^\mu_\nu - \delta^\mu_0 \delta^0_\nu$ is the projection operator onto spatial hypersurfaces of constant time in the background, and the harmonics $Q = Q(\mathbf{x}, \mathbf{k})$ are solutions of the scalar Helmholtz equation which are normalized according to equation (34) in the appendix. We assume that inflation is driven by a scalar field, which we call ψ in order to make a distinction with the field ϕ which was defined on a fixed geometry in the previous sections. The Lagrangian density of the field ψ is given by

$$\mathcal{L}_\psi = \frac{1}{8\pi G} R - \ g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \ - V(\psi), \tag{15}$$

where the potential term $V(\psi)$ is bounded from below. It will be useful to write the field ψ as the sum of a perturbation $\delta\psi(\mathbf{x}, t)$, and the mean value $\psi_0(t)$, which is defined as the average of ψ, evaluated on a spatial hypersurface of constant time in the background.

The mass of the field ψ is defined as the curvature of the potential, i.e. $m^2 := \frac{1}{2}\partial^2 V(\psi)/\partial\psi^2$, which is evaluated at $\psi = \psi_0$. Nearly exponential expansion occurs when the kinetic term in the Lagrangian (15) is small compared to the potential term (so that $T^\mu_\nu \approx \frac{1}{2}\delta^\mu_\nu V(\psi_0)$), and $V(\psi_0)$ is approximately constant on a timescale of the expansion time. It can be shown that nearly exponential expansion can occur for a large class of potentials, including potentials of the polynomial form $V(\psi) = \lambda\psi^{2n}$, with $\lambda \in R^+$ and $n \in \mathbf{Z}^+$ (see e.g. [3]).

In the following, we will denote perturbations by their Fourier components, which we defined in the appendix, and we drop the argument $\mathbf{k}$, unless there is a risk of confusion.

A gauge invariant variable which is useful in the description of perturbations is given by,

$$\phi_m := \delta\psi + \frac{\dot{\psi}_0}{H} \ (H_L + \frac{1}{3} H_T), \tag{16}$$

and ϕ_m can be interpreted in terms of the three-curvature of the constant-ψ hypersurfaces,

$$\mathcal{R}_{\text{constant}\psi} = \frac{4H}{\dot{\psi}_0} \ \frac{\mathbf{k}^2}{S^2} \phi_m, \tag{17}$$

where we assume that $\dot{\psi}_0 \neq 0$.

Copying Bardeen's notation, we define ϵ_m as the fractional energy density perturbation in the comoving time-orthogonal gauge, i.e.,

$$\epsilon_m := \delta T^0_0 / T^0_0, \tag{18}$$

evaluated in the gauge where $\delta\psi = B = 0$.

A variable related to ϵ_m is α_m, which is defined as the fractional lapse function perturbation, evaluated in the comoving time-orthogonal gauge, i.e. $\alpha_m := A$ in the gauge where $\delta\psi = B = 0$. From the expression for the energy

density of the scalar field in the comoving time-orthogonal gauge, i.e. $T^0_0 = \frac{1}{2}(g^{00}\dot{\psi}^2 + V(\psi))$, it follows that

$$\epsilon_m = -(1+\omega)\alpha_m, \tag{19}$$

where $\omega := -1 + 2/(1 + V(\psi)/\dot{\psi_0}^2)$. The fractional energy density perturbation ϵ_m is coupled to the entropy perturbation η_e, defined as the difference between the fractional isotropic pressure in the perturbed spacetime, and the fractional isotropic pressure at a point in the background spacetime with the same energy density. The entropy perturbation η_e, and the fractional lapse function perturbation α_m, can be shown to be proportional [7],

$$\alpha_m = \frac{\omega}{(1+\omega)(c_s^2 - 1)}\eta_e, \tag{20}$$

where we specialized Bardeen's equation (5.20) to the case with scalar field matter, and the speed of sound c_s^2 can be expressed in terms of background variables. Hence, ϵ_m, α_m and η_e all represent the same physical perturbation, but unlike ϵ_m and α_m, the interpretation of η_e is not related to the properties of some collection of spatial hypersurfaces in the perturbed spacetime, and it is therefore an intuitively clear measure of an intrinsic perturbation.

The gravitational and inflaton action has been expanded to second order in terms of the scaled perturbation variable $w := S\phi_m$ and background quantities, by Deruelle et al. [22]. Recall that the variable ϕ_m, defined by expression (16), is only to linear order invariant under a gauge transformation. The square of a first order gauge invariant variable, is however gauge invariant to second order. An expansion of the action which contains terms of linear and quadratic order in ϕ_m, is not in general gauge invariant to second order, unless the linear terms in the expansion vanish. Note that an expansion of the action in terms of the perturbation variable ϕ_m, is obtained by integrating the expansion of the Lagrangian (15) and the volume element, over the background spacetime. Indeed, terms which are linear in ϕ_m vanish in this integration, due to the orthogonality relation for the harmonics Q (equation (34) in the appendix).

The field equation for w appears to be of a simple Klein-Gordon type with a time dependent mass term,

$$(\frac{\partial^2}{\partial\eta^2} + \mathbf{k}^2 - 2\eta^{-2})w(\mathbf{k}) = 0, \tag{21}$$

which has the solutions

$$w(\mathbf{k}) = -\frac{\sqrt{-\pi\eta}}{4}H^{(1)}_{\frac{3}{2}}(-\eta\mathbf{k}), \tag{22}$$

where $H^{(1)}_q$ denotes the Hankel function of the first kind [24]. Note that equation (21) is derived from an expansion of the action of the inflaton field *and* the

geometry, in terms of the gauge invariant variable w [22]. The coupling between the inflaton and the geometry is therefore accounted for by the requirement that the solutions (22) extremize the combined gravitational and inflaton action.

The field operator which is associated with the second quantized variable w can be expanded in terms of the solutions (22), and the two-point correlation function for the field w takes the usual form [12],

$$G_0(\mathbf{k}) = \frac{1}{2}\ |w(\mathbf{k})|^2, \tag{23}$$

where we assumed that the field is in the vacuum state which is natural in comoving coordinates. In the following we assume that the quantum state of the w-field has decohered (see section 3), in which case expression (23) can be interpreted as the sum over decohered configurations $\{w\}$ of the square of the perturbation component $w(\mathbf{k})$ evaluated for $\{w\}$, multiplied by the probability of finding the configuration $\{w\}$.

The asymptotic behavior of $G_0(\mathbf{k})$ near $\mathbf{k} = 0$ follows from expression (22), (23) and the expansion of the Hankel function, $H_q^{(1)}(z) \sim z^{-q}$ for $z \downarrow 0$. We find

$$G_0(\mathbf{k}) \sim \eta^{-2}\mathbf{k}^{-3}, \tag{24}$$

for $\mathbf{k} \downarrow 0$ and η constant. Assuming decoherence, expression (24) describes the asymptotic behavior of the *classical* perturbation component $w(\mathbf{k})$ for $\mathbf{k} \downarrow 0$,

$$w(\mathbf{k}) \sim \eta^{-1}\mathbf{k}^{-\frac{3}{2}}, \tag{25}$$

where here and in the following we neglect a probabilistic factor of order one on the right-hand side of the $\sim$ symbol.

4.1 Energy and lapse function perturbations

Let us now determine the spectrum of fractional energy density and lapse function perturbations. The fractional energy density perturbation ϵ_m has been evaluated in terms of w in [23],

$$T^0_0\epsilon_m = 2\frac{\kappa H}{S^2}[\dot{w} - HSw], \tag{26}$$

where κ is a constant which depends on the background geometry. Using expression (22) for w, and the differentiation property of the Hankel function, $(\mathrm{d}/\mathrm{d}z)H_q(z) + (q/z)H_q(z) = H_{q-1}(z)$, we find,

$$T^0_0\epsilon_m = \frac{\kappa\sqrt{\pi}}{S^2}\ \mathbf{k}\sqrt{-\eta}\ H^{(1)}_{\frac{1}{2}}(-\mathbf{k}\eta), \tag{27}$$

where we used that the scale factor S equals $(H\eta)^{-1}$ in the case of exponential expansion. Using the asymptotic expression for the Hankel function, $H_q^{(1)}(z) \sim z^{-q}$ for $z \downarrow 0$, yields,

$$\epsilon_m \sim \alpha_m \sim \eta^2\mathbf{k}^{\frac{1}{2}}. \tag{28}$$

The time dependence in equation (28) is a consequence of the Fourier decomposition of the perturbations in comoving rather than physical k-space. Let $\bar{\epsilon}_m$ and $\bar{\alpha}_m$ denote the physical k-space Fourier components of the fractional energy density and lapse function perturbations, which we define by expression (40) in the appendix. According to expression (41) in the appendix, the perturbation components in physical and comoving k-space are related by $\bar{\epsilon}_m = S^{\frac{3}{2}}\epsilon_m$ and $\bar{\alpha}_m = S^{\frac{3}{2}}\alpha_m$. Using this relation, and expression (28), yields the asymptotic expression,

$$\bar{\epsilon}_m \sim \bar{\alpha}_m \sim k^{\frac{1}{2}}, \tag{29}$$

where $k = S^{-1}\mathbf{k}$ denotes the physical wavenumber, and there is no time dependence at the right-hand side of equation (29). Expression (29) shows that the infra-red behavior of the fractional density perturbation and lapse function is regular, and integrable about $k = 0$.

Note that expression (29) determines the asymptotic behavior of the spectrum of fractional energy density perturbations, but not its magnitude. Observations of perturbations at the time of last scattering [2], put an upper limit on ϵ_m of the order of 10^{-5} for wavenumbers small compared to the value of the Hubble parameter at that time. Since ϵ_m grows approximately as the square of the scale factor for the growing mode and small wavenumbers [7], the fractional energy density perturbations must have been many orders of magnitude smaller than 10^{-5} at the time when inflation came to an end.

4.2 Curvature perturbations

In this subsection, we investigate the spectrum of curvature perturbations of the hypersurfaces on which the inflaton field ψ is constant. According to expression (41) in the appendix, the perturbation components in physical and comoving k-space are related by $\bar{\phi}_m = S^{\frac{3}{2}}\phi_m$. Using expression (25), we find the asymptotic behavior of $\bar{\phi}_m$ for $k \downarrow 0$,

$$\bar{\phi}_m = S^{\frac{3}{2}}\phi_m \sim k^{-\frac{3}{2}}, \tag{30}$$

and there is no time dependence in equation (30).

From expressions (17) and (25) we derive the expression for the Fourier component of the spatial curvature perturbation, with respect to the physical wavenumber k,

$$\bar{\mathcal{R}}_{\text{constant}\psi} \sim k^{\frac{1}{2}}, \tag{31}$$

for $k \downarrow 0$.

We will now address the question whether the geometry of the constant-ψ hypersurfaces develops increasingly strong inhomogeneities at large length scales or at late times. More generally, one would like to know whether our perturbed FLRW spacetime remains everywhere close to FLRW in some well defined sense.

The question whether an inhomogeneous spacetime is close to FLRW in some objective sense, is known as the averaging problem in cosmology. This problem

appears to be surprisingly nontrivial, even in the linearized case, since it involves comparing tensors in different spacetimes, in a way which is independent of the choice of coordinates, gauge, and the averaging prescription (see e.g. [21] and references therein).

In the following we will sidestep the difficulties involved with a precise formulation of the averaging problem, by considering the scaling behavior of the spectrum of perturbations. Without presenting an objective criterion for a spacetime being close to an FLRW spacetime, we determine whether intrinsic perturbations grow or decrease under a change of the length scale. This allows us to compare the warping of our universe at superhorizon scales and at length scales where observations can be made, e.g., on the intersection of our past light cone with the surface of last scattering. If perturbations tend to grow at large length scales, then our perturbative treatment can be expected to break down at superhorizon scales, which is in agreement with the picture of a stochastically inflating universe which is strongly warped at superhorizon scales [3]- [5].

On the contrary, a decreasing or constant spectrum of perturbations implies that our perturbative approach holds on superhorizon scales, given that it holds at subhorizon scales. This will then give us an improved estimate of the effect of quantum fluctuations on the global structure of the universe.

We define a new length scale $\bar{\ell}$ by choosing a new unit of physical length which is $a \in \mathbf{R}$ times the former unit of length. In terms of the new length scale, a physical quantity q with dimension ℓ^d transforms to $\bar{q} = a^{-d} q$. As we derived in the appendix, the Fourier transform $q(k)$ of a quantity $q(x)$ transforms under scale transformations as a half-density of this quantity in k-space, i.e. $\bar{q}(\bar{k}) = a^{-d-\frac{3}{2}} q(k)$, where $\bar{k} = ak$.

The scaled spatial curvature perturbation $\bar{\mathcal{R}}(k)$ therefore takes the form

$$\bar{\mathcal{R}}_{\text{constant}\phi} = \ a^{\frac{1}{2}} \mathcal{R}_{\text{constant}\phi}, \tag{32}$$

where we used that the spatial curvature has the dimension ℓ^{-2}.

Using expression (31) for the spectrum of three-curvature perturbations, and the scale transformation property (32), we find,

$$\bar{\mathcal{R}}_{\text{constant}\phi} \sim a^{\frac{1}{2}} k^{\frac{1}{2}} \ = \bar{k}^{\frac{1}{2}}. \tag{33}$$

Comparing equations (31) and (33), we find that the spectrum of curvature perturbations of the constant-ψ hypersurfaces remains constant under a change of the length scale.

Summarizing the results derived in this subsection, we found that perturbations in the spatial curvature are regular about $k = 0$, and the perturbations do not grow at late times or at large length scales.

4.3 Interpretation

Let us now address the question why the gauge invariant perturbations $\bar{\mathcal{R}}$ and $\bar{\alpha}_m$ do not show the same infra-red divergence as we found for a quantum field ϕ

on a fixed geometry. At first sight, this might be surprising, since expressions (5) and (24) show that the gauge invariant perturbations ϕ_m, and the perturbations ϕ on a fixed geometry, diverge with the same power when $k \downarrow 0$. The essential difference between the perturbations ϕ_m and ϕ, is their interpretation in terms of physical quantities. Note that a perturbation of the field ϕ on a fixed geometry is proportional to a perturbation in the energy density, which follows by expanding the energy density $T^0_0 = \frac{1}{2}(\partial^0\phi\partial_0\phi + V(\phi))$ about the background value $\phi = \phi_0$, where we use that $V'(\phi)|_{\phi_0} \neq 0$ as long as the scalar field has not reached a minimum of the potential. On the contrary, it follows from equation (29) and (30) that the gauge invariant amplitude ϕ_m acts as a *potential* for the fractional lapse function and energy density perturbations in the limit when $k \downarrow 0$. Similarly, expression (17) shows that ϕ_m acts as a potential for the spatial curvature perturbation $\mathcal{R}$. The perturbation ϕ_m may therefore diverge as $k^{-\frac{3}{2}}$ for $k \downarrow 0$, while the spatial curvature $\mathcal{R}$, and the fractional lapse function and energy density perturbations α_m and ϵ_m are well behaved when $k \downarrow 0$.

One may question whether the time evolution of the perturbations in the spatial curvature and the fractional lapse function can be described by a diffusion equation of the form (12). For a massless quantum field ϕ on a fixed geometry, this method is natural since the expectation value of $|\phi|^2$, which according to equation (13) equals to the standard deviation of the probability distribution $P(\phi, t)$, appeared to grow linearly in time. In this section we showed that the spectrum of fractional lapse function and spatial curvature perturbations is integrable about $k = 0$, and constant in time. The probability distribution which describes the decohered fractional lapse function and spatial curvature perturbations, is therefore time independent, and it does not evolve according to the diffusion equation (12).

5 Conclusions

In the previous section we derived expressions for the spectrum of the fractional lapse function and spatial curvature perturbations which are generated during exponential expansion. We found that intrinsic perturbations of the scalar field and the geometry do not grow at late times or at large length scales, during exponential expansion. The constancy of the spectrum of perturbations in space and time, and the observational bounds on the magnitude of the perturbations, justify our perturbative approach. Our result contradicts the assumption that quantum fluctuations grow nonperturbatively during inflation, which underlies the idea of stochastic inflation. This indicates that an inflating universe is not likely to develop a highly irregular structure at superhorizon scales, unless these irregularities are present in the initial conditions.

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7 Appendix

In this appendix, we discuss the scaling behavior of the flat space Fourier transform. Let us define the solutions $Q(\mathbf{k}, \mathbf{x})$, which satisfy the scalar Helmholtz equation, i.e. $Q_{;i}^{;i} = |\mathbf{k}|^2 Q$, where $;i$ denotes the covariant derivative with respect to $\mathbf{x}^i$, and the solutions Q are normalized according to

$$\int \mathrm{d}^3\mathbf{x} Q^*(\mathbf{k}, \mathbf{x}) Q(\mathbf{k}', \mathbf{x}) \ = \delta^3(\mathbf{k} - \mathbf{k}'). \tag{34}$$

The Helmholtz equation and the normalization condition (34) are satisfied by distributions of the form $Q(\mathbf{k}, \mathbf{x}) = Ne^{i\mathbf{k}\mathbf{x}}$, where the normalization factor N is defined as the distribution which is constant for all $\mathbf{x}$, and which is normalized as,

$$\int \mathrm{d}^3\mathbf{x} N^2 = (2\pi)^{-3}. \tag{35}$$

The Fourier transform of a function $f(\mathbf{x})$ with respect to the comoving wavenumber $\mathbf{k}$ is given by,

$$f(\mathbf{k}) = \int \mathrm{d}^3\mathbf{x} Q(\mathbf{k}, \mathbf{x}) f(\mathbf{x}). \tag{36}$$

Let us now consider how $f(\mathbf{k})$ transforms under scale transformations. We implement a scale transformation by defining a new unit of length $\bar{\ell}$, which equals $a \in \mathbf{R}$ times the former unit of length. In terms of the new length scale, we define solutions $\bar{Q}$ of the Helmholtz equation, which are normalized by the condition

$$\int \mathrm{d}^3\bar{x} \bar{Q}^*(\bar{k}, \bar{x}) \bar{Q}(\bar{k}', \bar{x}) \ = \delta^3(\bar{k} - \bar{k}'). \tag{37}$$

A basis of solutions $\bar{Q}$ which satisfy the normalization condition (34) is given by $\bar{Q}(k, x) = \bar{N} e^{i\bar{k}\bar{x}}$, where $\bar{N}$ is defined as the constant distribution which satisfies the normalization condition

$$\int \mathrm{d}^3\bar{x} \bar{N}^2 = (2\pi)^{-3}. \tag{38}$$

It follows from the definitions (35) and (38) that the normalization constants N and $\bar{N}$ are related by $\bar{N} = a^{\frac{3}{2}} N$, which implies

$$\bar{Q}(\bar{k}, \bar{x}) = a^{\frac{3}{2}} Q(k, x), \tag{39}$$

where $\bar{x} = a^{-1}\mathbf{x}$ and $\bar{k} = a\mathbf{k}$. The Fourier transform of a function with respect to the barred wavenumber $\bar{k}$ is defined by

$$f(\bar{k}) := \int \mathrm{d}^3\bar{x}\ \bar{Q}(\bar{k}, \bar{x}) f(\bar{x}), \tag{40}$$

and from the relation (39) and the definitions (36) and (40) it follows that

$$f(\bar{k}) = a^{-\frac{3}{2}} f(\mathbf{k}), \tag{41}$$

where $\bar{k} = a\mathbf{k}$. Expression (41) shows that the Fourier transform of a function $f(x)$ transforms under a scale transformation as a half-density in k-space. In some calculations it will be useful to express the Fourier transform with respect to the comoving wavenumber (36), in terms of the Fourier transform with respect to the physical wavenumber. Since the comoving wavenumber and the physical wavenumber are related by a time dependent scale transformation $k = S^{-1}\mathbf{k}$, this is simply a special case of the scale transformation discussed above, and $f(k)$ and $f(\mathbf{k})$ are related by $f(k) = S^{\frac{3}{2}} f(\mathbf{k})$, where $k = S^{-1}\mathbf{k}$.

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